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# Group Representations in Certain Lattices of Propositions

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*Abstract*

Preparatory to a possible in-depth study of dynamical symmetries in quantal systems, this paper establishes the fundamental aspects of group representations in complete orthocomplemented weakly modular lattices (CROC's). Several related results, pertaining to the structure, direct sum decomposition, and invariance of CROC's, are also found.

## 1. *Introduction*

During the last decade considerable effort has been directed toward the abstract description of both quantal and classical systems in terms of suitable lattices of propositions. This approach has been originally suggested by the early work of Birkhoff and von Neumann (1936). A systematic review of the field can be found in Jauch (1968) (see also Varadarajan, 1968). The fundamental theorems are summarized in Piron's work (1964). A more recent survey, including newer developments, was given by Piron (1972a).

A particular attractive feature of this approach is that it may provide us with a deep insight into the origin and nature of dynamical symmetries. Progress in this direction was recently made by Piron (1972b; also Piron, 1972a), who motivated the emergence of the Galilei group and discussed its relation to gauge invariance (in this respect, see also Jauch, 1964).

It is well known that, in the customary framework of quantum theory, the derivation and application of symmetry principles is rendered possible by the fact that there exists a well developed theory of group representations in Hilbert spaces. On the other hand, no representation theory of groups in lattices corresponding to quantal proposition systems is known. We believe that significant progress in the understanding of the origin, motivation, and dynamical role of symmetries can be achieved only when such a representation theory is available.

The purpose of this paper is to study the foundations of a theory of group representations in complete, orthocompleted, weakly modular lattices. Apart from establishing the basic representation theorems, we also obtain some additional insight into certain properties (like invariance, decomposability) of the relevant lattices.

We do not consider here the specific problems associated with continuous representations of Lie groups. This study would necessitate the introduction of suitable topologies. This problem will be considered at a later time.

## 2. *General Preparations*

We begin by quoting, for the reader's convenience, three well-known definitions (cf. Piron, 1964; Loomis, 1955; and Maeda, 1955).

*Definition* 1 (*CROC*). Let $\mathscr{L}$ be a set equipped with a partial order relation $<$. Suppose that any arbitrary family $\{x_j\}$ of elements possesses an infimum

$$\bigwedge_j x_j$$

Suppose further that there is defined an orthocomplementation** $x \mapsto x'$ on $\mathscr{L}$. Finally suppose that $\mathscr{L}$ is weakly modular.† Then $\mathscr{L}$ will be called a complete orthocomplemented weakly modular lattice, or CROC‡ for short.

*Remarks.* (a) Completeness§ guarantees the existence of a first element, to be denoted by $O$. (b) Completeness and orthocomplementation guarantee the existence of a supremum

$$\bigvee_j x_j$$

for arbitrary families. Then the existence of a last element $I$ follows, and we also have $x \vee x' = I$.

*Definition* 2 (*Direct Union*). Let $\{\mathscr{L}_\alpha\}$ be an arbitrary family of CROC's. Consider the set $\mathscr{L}$ of all families $\{x_\alpha\}, x_\alpha \in \mathscr{L}_\alpha$, all $\alpha$. Define the order relation

$$\{x_\alpha\} < \{y_\alpha\} \text{ iff } x_\alpha < y_\alpha$$

for all $\alpha$, and the complementation

$$\{x_\alpha\}' = \{x_\alpha{}'\}$$

**An orthocomplementation is a decreasing involution $x \mapsto x'$ of order 2, so that $x'' = x$, $x \wedge x' = 0, x < y \Leftrightarrow y' < x'$.

† This means that if $x < y$ then $\{x, x', y, y'\}$ generates a Boolean lattice. Equivalently, $x < y$ implies $x \vee (x' \wedge y) = y$.

‡ CROC is an acronym for "Canoniquement relativement ortho-complementé."

§ I.e., the unrestricted existence of infima.

Then $\mathscr{L}$ is a CROC, called the direct union of the $\mathscr{L}_\alpha$'s, and we write

$$\mathscr{L} = \bigvee_\alpha \mathscr{L}_\alpha$$

*Definition* 3 (*Compatability*). Two elements of a CROC are said to be compatible (in symbols: $x \leftrightarrow y$) if any one of the following equivalent conditions** is satisfied:

i. $(x \wedge y') \vee y > x$ or, dually, $(x \vee y') \wedge y < x$
ii. Any possible (nontrivial) distributivity relation between $x, y, x', y'$ holds true
iii. $(x \wedge y) \vee (x \wedge y') \vee (x' \wedge y) \vee (x' \wedge y') = I$

*Remarks.* (a) It follows that $x \leftrightarrow y$ iff $x \leftrightarrow y'$. (b) A lattice is weakly modular iff $x < y$ implies $x \leftrightarrow y$.

We now introduce and study a crucial new concept, to be called star map.**

*Definition* 4 (*Star map*). Let $\mathscr{L}$ be a CROC, and let $\mathscr{M}$ be a subset of $\mathscr{L}$ which is a complete lattice with respect to the operations $\wedge$, $\vee$ of $\mathscr{L}$. The star map*: $\mathscr{M} \to \mathscr{L}$ is defined by

$$x \mapsto x^* \equiv (x' \vee a) \wedge b$$

where $a(b)$ is the first (last) element of $\mathscr{M}$.

*Lemma* 1. Let $\mathscr{M} \subset \mathscr{L}$ be as specified in Def. 4 and assume that $\mathscr{M}$ is closed under the star map. Then * is an orthocomplementation on $\mathscr{M}$.

*Proof.*

α.

$$(x^*)^* = [(a \vee x') \wedge b]^* = \{[(a \vee x') \wedge b]' \vee a\} \wedge b$$
$$= [(a' \wedge x) \vee b' \vee a] \wedge b = (x \vee b') \wedge b = x$$

β. Let $x, y \in \mathscr{M}$ so that

$$y^* = (y' \vee a) \wedge b,\ x^* = (x' \vee a) \wedge b$$

Since $x < y$ iff $y' < x'$, we see that, by isotonity, $x < y$ implies $y^* < x^*$. Conversely, since $x^* < y^* \Rightarrow (y^*)^* < (x^*)^*$, we have by (α) that $x^* < y^*$ implies $y < x$.

γ.

$$x \wedge x^* = x \wedge b \wedge (a \vee x') = x \wedge (a \vee x') = (x \wedge a) \vee (x \wedge x') = a$$

where we used the fact that $x \leftrightarrow a \leftrightarrow x'$ and took notice of Remark (*b*) following Def. 3, which then permits distributivity.

** Cf. Piron (1964) p. 454.
† Such a map, but restricted to segments only, has been studied already by Piron (1964).

*Lemma* 2. Let $\mathscr{M} \subset \mathscr{L}$ be as specified in Def. 4 and assume that $\mathscr{M}$ is closed under the star map. Then $\mathscr{M}$ is a CROC with the operations $\Lambda$, V, $^*$.

*Proof.* It suffices to show weak modularity (cf.† footnote, p. 74). If $x < y$, then by the remarks following Def. 3 we can use compatibility and hence distributivity to calculate

$$x\mathrm{V}(x^*\Lambda y) = x\mathrm{V}\{[(x'\mathrm{V}a)\Lambda b]'\Lambda y\} = x\mathrm{V}[(x'\mathrm{V}a)\Lambda y]$$
$$= x\mathrm{V}[(x'\Lambda y)\mathrm{V}(a\Lambda y)] = x\mathrm{V}(x'\Lambda y) = y$$

where in the last step, weak modularity of $\mathscr{L}$ was used.

In view of this lemma, we are led to the following:

*Definition* 5 *(Subcroc)*. If $(\mathscr{L}, \Lambda, \mathrm{V}, ')$ is a CROC and $\mathscr{M}$ is a subset of $\mathscr{L}$ which is a complete lattice relative to $\Lambda$, and V, and if $\mathscr{M}$ is closed under the star map, then we call $(\mathscr{M}, \Lambda, \mathrm{V}, *)$ a subcroc of $\mathscr{L}$.**

As an extension of Piron's Theorem X1 (Piron, 1964), we now establish the following:

*Lemma* 3. Let $\mathscr{M}$ be a subcroc of $\mathscr{L}$. Then $x \leftrightarrow y$ in $\mathscr{M}$ iff $x \leftrightarrow y$ in $\mathscr{L}$.

*Proof.* Trivially, for $x, y \in \mathscr{M}$, we have $a < x, y < b$ and $x \leftrightarrow a \leftrightarrow y$, so that we can write

$$x\mathrm{V}(x^*\Lambda y) = x\mathrm{V}[(a\mathrm{V}x')\Lambda b\Lambda y] = x\mathrm{V}[(y\Lambda a)\mathrm{V}(y\Lambda x')] = x\mathrm{V}(x'\Lambda y)$$

from which our statement follows via (i) of Def. 3.

We conclude this section by studying intersections of subcrocs.

*Lemma* 4. The set theoretic intersection of two subcrocs is a subcroc.

*Proof.* If $\mathscr{M}_1$ and $\mathscr{M}_2$ are subcrocs of $\mathscr{L}$, the $\mathscr{M}_1 \cap \mathscr{M}_2$ is a complete sublattice of $\mathscr{L}$. Denoting the first (last) element of $\mathscr{M}_1 \cap \mathscr{M}_2$ by $a(b)$ respectively, we have

$$x^* \equiv (x'\mathrm{V}a)\Lambda b = (x'\Lambda b)\mathrm{V}a \tag{2.1}$$

(The second equality follows from the fact that, since $a < x < b$, the elements $x, a, b, x'$ are pairwise compatible, so that distributivity can be invoked, and furthermore $a\Lambda b = a$.) Denoting the first (last) elements of $\mathscr{M}_k$ $(k = 1, 2)$ by $a_k(b_k)$ and the corresponding star maps by $*1(*2)$ respectively, from (2.1) we get (since $a_1 < a < b < b_1$)

$$\begin{aligned} x^* &= (x'\Lambda b\Lambda b_1)\mathrm{V}a = [(x'\Lambda b_1)\Lambda b]\,\mathrm{V}a = [(x'\Lambda b_1)\mathrm{V}a]\,\Lambda b \\ &= [(x'\Lambda b_1)\mathrm{V}a_1\mathrm{V}a]\,\Lambda b = (x^{*1}\mathrm{V}a)\Lambda b \in \mathscr{M}_1 \end{aligned} \tag{2.2}$$

A similar reasoning gives

$$x^* = (x^{*2}\mathrm{V}a)\Lambda b \in \mathscr{M}_2 \tag{2.3}$$

From (2.2) and (2.3) we get $x^* \in \mathscr{M}_1 \cap \mathscr{M}_2$, QED.

** Subcrocs that are arbitrary singleton subsets will be disregarded in the following.

## 3. *Morphisms, Invariance, and Decompositions*

We now turn our attention to questions relating to mappings between CROC's and associated structural properties. In a sense, these topics run parallel to similar questions relevant for Hilbert spaces and prepare the ground for the representation theory of groups. Some results of this section will not be needed in the sequel, but they are interesting per se.

To facilitate the discussion, we employ the standard definition of *orthogonality* of CROC elements: $x \perp y$ iff $x < y'$.

Following Piron (1971), we adopt the following specification of morphisms:

*Definition* 6 (*Morphism*). A morphism between two CROC's $\mathscr{L}_1$ and $\mathscr{L}_2$ is a map

$$\phi : \mathscr{L}_1 \rightarrow \mathscr{L}_2, \qquad x \mapsto \phi x$$

with the following properties:

i. $\phi \bigvee_j x_j = \bigvee_j \phi x_j$

ii. $x \perp y \Rightarrow \phi x \perp \phi y$

In particular, if $\phi$ is bijective, we speak of an isomorphism of CROC's and write $\mathscr{L}_1 \approx \mathscr{L}_2$. If $\mathscr{L}_1 \equiv \mathscr{L}_2 \equiv \mathscr{L}$ and $\phi$ is bijective, we call it an automorphism of $\mathscr{L}$.

*Remarks.* (a) It can be shown that a morphism has the following additional properties:

$$\phi O_1 = O_2, \qquad \phi \bigwedge_j x_j = \bigwedge_j \phi x_j, \qquad \phi x' = (\phi x)' \wedge \phi I_1$$

However, $\phi I_1 \neq I_2$, unless $\phi$ is an isomorphism. In that case, the third property simplifies to $\phi x' = (\phi x)'$. Also note that, rather trivially, $x < y$ implies $\phi x < \phi y$ for any morphism (and $\phi$ is strictly monotone for isomorphisms).

(b) Let $\Gamma$ be the set of all automorphisms of a CROC $\mathscr{L}$, equipped with the binary operation of composition of morphisms. In the familiar elementary manner it then follows that $\Gamma$ is a group, which will be called the *group of automorphisms* of $\mathscr{L}$.

Next we introduce the concept of invariance in an obvious manner:

*Definition* 7 (*Invariance*). Let $\mathscr{M}$ be a subcroc of $\mathscr{L}$ and let $\phi$ be a morphism of $\mathscr{L}$. We say that $\mathscr{M}$ is invariant under $\phi$ iff $m \in \mathscr{M}$ implies $\phi m \in \mathscr{M}$.

We can now establish the following:

*Lemma* 5. Let $\mathscr{M}$ be invariant under $\phi$ and suppose $\phi$ is actually an automorphism of $\mathscr{L}$. Then its restriction $\phi_{\mathscr{M}} : \mathscr{M} \rightarrow \mathscr{M}$ given by $\phi_{\mathscr{M}} x = \phi x$ is an automorphism of $\mathscr{M}$.

*Proof.* The only nontrivial fact that must be shown is that $x < y^*$ implies $\phi_{\mathscr{M}} x < (\phi_{\mathscr{M}} y)^*$. But $x < y^*$ means $x < b \wedge (y' \vee a)$, so that $\phi_{\mathscr{M}} x = \phi x < \phi b \wedge (\phi y' \vee \phi a)$, hence $\phi_{\mathscr{M}} x < b \wedge [(\phi y)' \vee a] = (\phi y)^*$, QED.

In the following two lemmas we want to study CROC's that are direct unions of subcrocs. So as to avoid confusion with later developments we agree to say that $\mathscr{L}$ is *c-irreducible* iff its center** $\mathscr{C}$ consists of $O$ and $I$ only. In the opposite case we shall say that $\mathscr{L}$ is *c-reducible.*

*Lemma* 6. A CROC is *c-reducible* iff it is isomorphic to the direct union of some of its subcrocs.

*Proof.* (a) Assume that $\mathscr{L}$ is $c$-reducible, i.e., that there exists some $z \in \mathscr{C}$ such that $z \neq 0$, $z \neq I$. Define†

$$\mathscr{M}_1 = \{z \Lambda x \equiv x_1 \mid x \in \mathscr{L}\} = [0, z]$$

$$\mathscr{M}_2 = \{z' \Lambda x \equiv x_2 \mid x \in \mathscr{L}\} = [0, z']$$

$\mathscr{M}_1$ and $\mathscr{M}_2$ are obviously subcrocs of $\mathscr{L}$. Furthermore, for any $x \in \mathscr{L}$, $x = x_1 \mathrm{V} x_2$. Finally $x < y$ implies $x_i < y_i$ $(i = 1, 2)$, so that indeed $\mathscr{L} \approx \mathscr{M}_1 \mathrm{V} \mathscr{M}_2$.

(b) Assume $\mathscr{L} \approx \mathrm{V}_\alpha \mathscr{M}_\alpha$, where each $\mathscr{M}_\alpha$ is a subcroc. Consider the family $\hat{x}_\alpha = \{x_j\}$, where

$$x_j = \begin{cases} b_\alpha & \text{if } j = \alpha \\ a_\alpha & \text{if } j \neq \alpha \end{cases}$$

If now $\{y_\alpha\}$is an arbitrary element of $\mathrm{V}_\alpha \mathscr{M}_\alpha$, we trivially have

$$(\{y_\alpha\}' \Lambda \hat{x}_\alpha) \mathrm{V} \{y_\alpha\} > \hat{x}_\alpha$$

hence $\hat{x}_\alpha$ is in the center of $\mathrm{V}_\alpha \mathscr{M}_\alpha$, hence $\mathscr{L}$ is $c$-reducible.

*Lemma* 7. Let $\mathscr{L} \approx \mathrm{V}_\alpha \mathscr{M}_\alpha$ where each $\mathscr{M}_\alpha$ is a subcroc of $\mathscr{L}$. Assume that for any pair $\alpha \neq \beta$ we have $\mathscr{M}_\alpha \perp \mathscr{M}_\beta$, by which we mean that $x_\alpha < x'_\beta$ for any $\alpha \neq \beta$. Assume further that either

i. At least one $\mathscr{M}_\alpha$ is not isomorphic to a segment of $\mathscr{L}$; or
ii. the last elements of the subcrocs $\mathscr{M}_\alpha$ are such that $\mathrm{V}_\alpha b_\alpha \neq I$ with each $b_\alpha$ being in the center of $\mathscr{L}$. Under these assumptions, $\mathscr{L}$ is infinite.

*Proof.* Since a set is infinite if it is isomorphic to one of its proper subsets, it will suffice to show that, under the conditions stated $\mathscr{L}$ is isomorphic to one of its subcrocs. To this end, we define the map

$$f : \mathop{\mathrm{V}}_\alpha \mathscr{M}_\alpha \to \mathscr{L}$$

given by

$$f(\{x_\alpha\}) = \mathop{\mathrm{V}}_\alpha x_\alpha$$

** The center of a CROC is the subset of elements that are compatible with all elements of $\mathscr{L}$.

† A segment $[n, m]$ of $\mathscr{L}$ is the set of all elements $x$ such that $n < x < m$.

(a) $f$ is injective. To see this, note first that, since $\mathcal{M}_\alpha \perp \mathcal{M}_\beta$, we have $\mathcal{M}_\alpha \leftrightarrow \mathcal{M}_\beta$, hence

$$b_\beta \Lambda (\underset{\alpha}{\mathrm{V}} x_\alpha) = x_\beta$$

so that

$$y_\beta \Lambda (\underset{\alpha}{\mathrm{V}} x_\alpha) = x_\beta \Lambda y_\beta$$

for any $y_\beta$. Now assume that

$$f(\{x_\alpha\}) = f(\{y_\alpha\})$$

i.e.,

$$\underset{\alpha}{\mathrm{V}} x_\alpha = \underset{\alpha}{\mathrm{V}} y_\alpha$$

Combining this with our preceding observation, we obtain

$$y_\beta = y_\beta \Lambda (\underset{\alpha}{\mathrm{V}} y_\alpha) = y_\beta \Lambda (\underset{\alpha}{\mathrm{V}} x_\alpha) = x_\beta \Lambda y_\beta$$

that is, $y_\beta < x_\beta$. On the other hand,

$$x_\beta = x_\beta \Lambda (\underset{\alpha}{\mathrm{V}} x_\alpha) = x_\beta \Lambda (\underset{\alpha}{\mathrm{V}} y_\alpha) = x_\beta \Lambda y_\beta$$

so that $x_\beta < y_\beta$. Thus, $x_\beta = y_\beta$, hence $\{x_\alpha\} = \{y_\alpha\}$.

(b) $f$ is monotone. Indeed, $\{x_\alpha\} < \{y_\alpha\}$ trivially implies $\mathrm{V}_\alpha x_\alpha < \mathrm{V}_\alpha y_\alpha$.

(c) The image of $f$, to be denoted in the sequel by $\mathcal{L}^+$, is a subcroc of $\mathcal{L}$. To see this, note first that, since $f$ is strictly monotone and $\mathrm{V}_\alpha \mathcal{M}_\alpha$ is a lattice, $\mathcal{L}^+$ is also a lattice with first and last element $\mathrm{V}_\alpha a_\alpha$ and $\mathrm{V}_\alpha b_\alpha$, respectively. If now $x \in \mathcal{L}^+$, we have

$$x^* \equiv [x' \mathrm{V} (\underset{\alpha}{\mathrm{V}} a_\alpha)] \Lambda (\underset{\alpha}{\mathrm{V}} b_\alpha) = [(\underset{\alpha}{\Lambda} x'_\alpha) \mathrm{V} (\underset{\gamma}{\mathrm{V}} a_\gamma)] \Lambda (\underset{\beta}{\mathrm{V}} b_\beta) = [\underset{\alpha}{\Lambda} (x'_\alpha \mathrm{V} a_\alpha)] \Lambda (\underset{\beta}{\mathrm{V}} b_\beta)$$
$$= [\underset{\alpha}{\mathrm{V}} (x'_\alpha \Lambda b_\alpha) \mathrm{V} a_\alpha] \equiv \underset{\alpha}{\mathrm{V}} x^{*\alpha} \in \mathcal{L}^+$$

which was to be shown.

We now know that, by construction, the map

$$f : \underset{\alpha}{\mathrm{V}} \mathcal{M}_\alpha \to \mathcal{L}^+$$

is bijective and monotone. We further observe that

i. $f(\underset{k}{\mathrm{V}} \{x_\alpha{}^k\}) = \underset{\alpha}{\mathrm{V}} \underset{k}{\mathrm{V}} x_\alpha{}^k = \underset{k}{\mathrm{V}} f(\{x_\alpha{}^k\})$

and

ii. If $\{x_\alpha\} \perp \{y_\alpha\}$, then $\mathrm{V}_\alpha x_\alpha < \mathrm{V}_\alpha y_\alpha^{*\alpha}$, which means that $f(\{x_\alpha\}) \perp [f(\{y_\alpha\})]^*$.

In summary we see that the map $f$ onto $\mathscr{L}^+$ is an isomorphism. Now, by the basic assumption of the lemma, we also have a CROC isomorphism $\phi: \mathscr{L} \to \mathrm{V}_\alpha \mathscr{M}_\alpha$ so that

$$f \circ \phi : \mathscr{L} \to \mathscr{L}^+$$

is an isomorphism. Suppose we have $\mathscr{L}^+ \equiv \mathscr{L}$, and let $x \in \mathscr{L}$. Then $x = \mathrm{V}_\alpha x_\alpha$ and $x_\alpha = x \Lambda b_\alpha$, so that $\mathscr{M}_\alpha = [0, b_\alpha]$ for each $\alpha$. We trivially also get that $I = \mathrm{V}_\alpha b_\alpha$.

Conversely, assume that $\mathscr{M}_\alpha = [0, b_\alpha]$ for each $\alpha$ and that $\mathrm{V}_\alpha b_\alpha = I$, where each $b_\alpha$ is in the center of $\mathscr{L}$. Then, since $\mathscr{M}_\alpha \perp \mathscr{M}_\beta$, we see that for any $x \in \mathscr{L}$, $x \Lambda b_\alpha = x_\alpha$, and also $\mathrm{V}_\alpha x_\alpha = \mathrm{V}_\alpha (x \Lambda b_\alpha) = (x \Lambda \mathrm{V}_\alpha b_\alpha) = x$. From these observations follows that the map $\phi$ can be characterized as follows: $\phi x = \{b_\alpha \Lambda x\} = \{x_\alpha\}$. Thus, $f = \phi^{-1}$, so that $\mathscr{L} \equiv \mathscr{L}^+$.

In summary, we see that $\mathscr{L} \equiv \mathscr{L}^+$ iff both $\mathscr{M}_\alpha = [0, b_\alpha]$ for each $\alpha$, and $\mathrm{V}_\alpha b_\alpha = I$. Therefore, if at least one of the conditions (i) and (ii) of the lemma is not satisfied, then $\mathscr{L}^+$ is a *proper* subcroc of $\mathscr{L}$. This concludes the proof.

We may supplement the preceding lemma by showing how, under a suitable additional condition, $\mathscr{L}^+$ can be actually constructed. But first we need the following:

*Definition* 8 (*Generated Subcroc*). Let $S$ be an arbitrary subset of $\mathscr{L}$. Consider all subcrocs $\mathscr{M}_j$ of $\mathscr{L}$ which contain $S$. Then $\cap_j \mathscr{M}_j \equiv \mathscr{M}(S)$ is a subcroc** and will be called the subcroc generated by $S$.

Now we have

*Lemma* 8. If the conditions of Lemma 7 hold and if in addition $a_\alpha = O$ for all $\alpha$, then $\mathscr{L}^+ = \mathscr{M}(S)$, where $S = \cup_\alpha \mathscr{M}_\alpha$.

*Proof.* Obviously $S \subset \mathscr{L}^+$. By definition, $\mathscr{M}(S) = \cap_j \mathscr{M}_j$, $S \subset \mathscr{M}_j$ for all $j$, so that $\mathscr{L}^+$ is one of the $\mathscr{M}_j$. On the other hand, if $x \in \mathscr{L}^+$, then $x = \mathrm{V}_\alpha x_\alpha$ so that $x \in \mathscr{M}_j$ for all $j$. Therefore $\mathscr{L}^+ \subset \mathscr{M}_j$ for all $j$. Thus, $\mathscr{L}^+ = \cap_j \mathscr{M}_j$, QED.

The purpose of the next lemma is merely to present a weaker criterion for the decomposition of a CROC into $c$-irreducible parts than the one which was given by Piron (1964). We have the following:

*Lemma* 9. A CROC is a direct union of orthogonal segments $\mathscr{L}_\alpha = [0, b_\alpha]$ such that $x = \mathrm{V}_\alpha x_\alpha$ and $x_\alpha = b_\alpha \Lambda x$ iff the center of $\mathscr{L}$ is a Boolean atomic CROC obeying the covering law† whose atoms $b_\alpha$ are disjoint, i.e., $b_\alpha \perp b_\beta$.

We relegate the rather technical proof to Appendix A.

Next we introduce a concept which is crucial for the representation theory.

*Definition* 9 (*Direct Sum*). Let $\mathscr{L}$ be a CROC and let $\{\mathscr{M}_\alpha\}$ be a class of subcrocs. Let

$$\phi : \mathscr{L} \to \underset{\alpha}{\mathrm{V}} \mathscr{M}_\alpha$$

** This follows from the generalization of Lemma 4.

† For the definitions of atomicity and the covering law, cf. Piron (1964).

be an isomorphism. Let $t : \mathscr{L} \to \mathscr{L}$ be an automorphism which leaves all $\mathscr{M}_\alpha$ invariant, i.e., $t\mathscr{M}_\alpha = \mathscr{M}_\alpha$ for each $\alpha$. Let $t_\alpha$ denote the restriction of $t$ to $\mathscr{M}_\alpha$. Define the map**

$$\{t_\alpha\} : \bigvee_\alpha \mathscr{M}_\alpha \to \bigvee_\alpha \mathscr{M}_\alpha$$

given by

$$\{x_\alpha\} \mapsto \{t_\alpha x_\alpha\}$$

Suppose that for each $t$ and each $\{t_\alpha\}$ we have the "commutation law"

$$\phi t = \{t_\alpha\}\phi$$

If such a map $\phi$ exists, then we say that $\mathscr{L}$ is the direct sum of the subcrocs $\mathscr{M}_\alpha$ and we write $\mathscr{L} = \oplus_\alpha \mathscr{M}_\alpha$.

Since the commutation law is clearly a very serious restriction on $\phi$, it is worthwhile to demonstrate on an explicit example that a nontrivial $\phi$ can be found. This will be done in Appendix B.

## 4. *Theorems on Representations*

We start with the following natural definition:

*Definition* 10 (*Representation*). Let $G$ be an abstract group with elements $g$. Let $R$ be a subgroup of the group $\Gamma$ of automorphisms of a CROC $\mathscr{L}$. If $A : G \to R$ is a (surjective) homomorphism, then we shall say that $A$ is a representation of $G$ in $\mathscr{L}$, and write $(G, A, \mathscr{L})$ or simply $(A, \mathscr{L})$. If $A$ is an isomorphism, the representation is said to be faithful.

Equally natural is the following:

*Definition* 11 (*Decomposability*). Let $\mathscr{L}$ be a CROC and $\mathscr{M}$ a nontrivial† subcroc which is invariant under all elements $r$ of a subgroup $R$ of the group of automorphisms $\Gamma$ of $\mathscr{L}$. Then we say that $R$ is a *reducible system* of automorphisms of $\mathscr{L}$. Otherwise $R$ is *irreducible*. Suppose that $\mathscr{L} = \oplus_\alpha \mathscr{M}_\alpha$, where each $\mathscr{M}_\alpha$ is invariant under $R$. Then we say that $R$ is *decomposable*. If $R$ is decomposable and each $\mathscr{M}_\alpha$ is $R$-irreducible, then $R$ is said to be *fully decomposable.*

Obviously, a $(G, A, \mathscr{L})$ will be called reducible (irreducible) depending on whether $R$ (corresponding to $A$) is such. Decomposability of a representation is defined accordingly.

We now state

*Theorem* 4.1 (*Schur's Lemma*). Let $G$ be a set of transformations and let $\mathscr{L}_1$ and $\mathscr{L}_2$ be two CROC's. Suppose that to each $g \in G$ there is associated an

** The map $\{t_\alpha\}$ is obviously an automorphism of $\mathrm{V}_\alpha \mathscr{M}_\alpha$.

† That is, $\mathscr{M} \neq \{O, I\}$.

automorphism $A(g)$ of $\mathscr{L}_1$ and an automorphism $B(g)$ of $\mathscr{L}_2$. If there exists a morphism $T: \mathscr{L}_1 \to \mathscr{L}_2$ with the property that $TA(g)x = B(g)Tx$ for all $x \in \mathscr{L}_1$ and every $g \in G$, then**

i. Ker $T$ is an invariant subcroc of $\mathscr{L}_1$,
ii. Im $T$ is an invariant subcroc of $\mathscr{L}_2$.

*Proof.* (i) We first show that Ker $T$ is a segment of $\mathscr{L}_1$. Obviously, $O_1 \in$ Ker $T$. Define $b = \vee x \in \mathscr{L}_1$, where the supremum $\vee$ is taken over all elements of Ker $T$. Clearly, $Tb = O_2$. Take now any $x \in \mathscr{L}_1$ such that $x < b$. Then $Tx < Tb = O_2$, hence $x \in$ Ker $T$. Thus, Ker $T = [O_1, b]$ as claimed. Since any segment is a subcroc, Ker $T$ is a subcroc. To show that it is invariant under $A(g)$, let $x \in$ Ker $T$, and then $TA(g)x = B(g)Tx = B(g)O_2 = O_2$. Thus $A(g)x \in$ Ker $T$ for all $g \in G$, QED.

(ii) Since, from the basic properties of a morphism, it easily follows that the image of a CROC is always a subcroc, we only need to show the invariance of Im $T$ under $B(g)$. If $y \equiv Tx \in$ Im $T$, then $B(g)y = B(g)Tx = TA(g)x \in$ Im $T$, QED.

Before formulating the next, closely related theorem, it will be useful to introduce the following definition:

*Definition* 12 (*Linearity*). An automorphism $T$ of a CROC $\mathscr{L}$ is said to be linear if it has a fixed point, i.e., if there exists an $x_0 \in \mathscr{L}(x_0 \neq 0, I)$ such that $Tx_0 = x_0$. Correspondingly, a representation $(G, A, \mathscr{L})$ of a group will be called a linear representation if all $A(g)$ are linear automorphisms of $\mathscr{L}$.

This terminology is motivated by comparison with (compact) linear operators on Hilbert spaces which always have an eigenray, $Tx_0 = \lambda x_0$. Some basic properties of linear automorphisms will be given in Appendix C.

We now state

*Theorem* 4.2. Let $A(g)$ be an irreducible system of automorphisms of $\mathscr{L}$. Then the only linear automorphism $T$ which commutes with all $A(g)$ is the trivial automorphism.

*Proof.* If $Tx_0 = x_0$ and $TA(g) = A(g)T$, then $TA(g)x_0 = A(g)x_0$. Since the set of all fixed points of $T$ is easily seen to be a subcroc, we observe that in fact it is now invariant under $A(g)$. Therefore, it is either the subcroc $\{O, I\}$ or $\mathscr{L}$ itself. But since we have at least one $x_0 \neq O, I$, the set of all fixed points of $T$ is $\mathscr{L}$ itself, i.e., $T = 1_{\mathscr{L}}$.

An elementary consequence is

*Theorem* 4.3. If $G$ is an Abelian group, then all its irreducible linear representations $(G, A, \mathscr{L})$ are one dimensional†.

** In an obvious way we define

$$\text{Ker } T = \{x \mid x \in \mathscr{L}_1 \text{ and } Tx = O_2\},$$
$$\text{Im } T = \{y \mid y \in \mathscr{L}_2 \text{ and } y = Tx \text{ for some } x \in \mathscr{L}_1\}.$$

† The dimension of a lattice is customarily defined as the supremum of the length of all segments that it contains.

*Proof.* Under the stated condition, for every given $A(g)$ one has $A(g)A(h) = A(h)A(g)$ for any $h$; therefore, by Theorem 4.2, $A(h)$ is the trivial automorphism $1_{\mathscr{L}}$, for all $h \in G$. Assuming that the CROC $\mathscr{L}$ is nontrivial (i.e., it has more than two elements), it will contain a subcroc $\{O, x\}$ with $x \neq I$, which is invariant under $1_{\mathscr{L}}$, contradicting the irreducibility of the representation.

The next theorem provides a criterion for reducibility of representations.

*Theorem* 4.4. Let $\mathscr{L}$ be a $c$-reducible CROC (with more than four elements) and suppose that it carries a representation $A$ of a group $G$. Then $A$ is reducible.

*Proof.* Our strategy will be to show that the center $\mathscr{C}$ of $\mathscr{L}$ is a subcroc invariant under $A$. Because of $c$-reducibility, there exists some $z \in \mathscr{C}$ such that $z \neq O, z \neq I$. By definition of $\mathscr{C}$, we also have $z' \in \mathscr{C}$ (and of course $O, I \in \mathscr{C}$). $\mathscr{C}$ is obviously a poset and in fact a complete sublattice, since, quite generally,

$$y_j \leftrightarrow x \text{ implies} \begin{cases} \bigvee_j y_j \leftrightarrow x \\ \\ \bigwedge_j y_j \leftrightarrow x \end{cases}$$

Furthermore, $z^* = z' \wedge I = z' \in \mathscr{C}$. Thus, $\mathscr{C}$ is actually a subcroc. If now $z \in \mathscr{C}$, then for any $y \in \mathscr{L}$,

$$(z \wedge y') \vee y > z$$

so that [cf. Remark (a) following Def. 6]

$$\{[A(g)z] \wedge [A(g)y]'\} \vee [A(g)y] > A(g)z$$

i.e., $A(g)z \leftrightarrow A(g)y$. Since $A(g)$ is bijective and $y$ was arbitrary, it follows that $A(g)z \in \mathscr{C}$, hence the subcroc $\mathscr{C}$ is invariant under $A$, QED.

Our next theorem permits the construction of $A$-invariant subcrocs from $A$-invariant subsets.

*Theorem* 4.5. Let $S \subset \mathscr{L}$ be a subset invariant under a system $A$ of automorphisms. Then the generated subcroc $\mathscr{L}(S)$ is invariant under $A$.

*Proof.* Because of Def. 8, $S \subset \mathscr{M}_j$ for each $j$, where $\cap_j \mathscr{M}_j = \mathscr{M}(S)$. Therefore, $A(g)S \subset A(g)\mathscr{M}_j$, so that, by the assumption of the theorem,

$$S \subset A(g)\mathscr{M}_j \tag{4.1}$$

Since $A(g)$ is an automorphism, $A(g)\mathscr{M}_j$ is a subcroc, and furthermore

$$A(g)\mathscr{M}_j \neq A(g)\mathscr{M}_{j'} \qquad \text{if } j \neq j'$$

Consider now the map

$$\{\mathscr{M}_j\} \to \{A(g)\mathscr{M}_j\}$$

given by

$$\mathscr{M}_j \mapsto A(g)\mathscr{M}_j$$

This is certainly surjective, and, by the preceding remark, it is also injective. Consequently [from (4.1)],

$$\{\mathscr{M}_j\} \equiv \{A(g)\mathscr{M}_j\} \tag{4.2}$$

because by definition $\{\mathscr{M}_j\}$ is the maximal family that contains $S$.

If now $x \in \mathscr{M}(S) = \cap_j \mathscr{M}_j$, then $x \in \mathscr{M}_j$ for all $j$, and so, by (4.2), $A(g)x \in A(g)\mathscr{M}_j$. Hence $A(g)x \in \cap_j A(g)\mathscr{M}_j = \cap_j \mathscr{M}_j = \mathscr{M}(S)$, so that $\mathscr{M}(S)$ is invariant under $A$, QED.

Our final theorem concerns the uniqueness of the decomposition of a representation. To facilitate the discussion, we first define the concept of equivalence:

*Definition* 13 (*Equivalence*). Let $\mathscr{L}_1$ and $\mathscr{L}_2$ be two CROC's. Let $\psi_1(\psi_2)$ be an automorphism of $\mathscr{L}_1(\mathscr{L}_2)$. Suppose there exists a morphism $T$: $\mathscr{L}_1 \to \mathscr{L}_2$ such that

$$T\psi_1 x = \psi_2 Tx$$

for all $x \in \mathscr{L}_1$. Then we say that $\psi_1$ and $\psi_2$ are similar. If, in particular, $T$ is an isomorphism, we say that $\psi_1$ and $\psi_2$ are equivalent automorphisms. Let now $\{\psi_1{}^s\}$ and $\{\psi_2{}^s\}$ be two systems of automorphisms of $\mathscr{L}_1$ and $\mathscr{L}_2$, respectively. If, for each $s$, $\psi_1{}^s$ is similar (equivalent) to $\psi_2{}^s$, we will say that the two systems of automorphisms are similar (equivalent). In case of equivalence we shall write $(\psi_1{}^s, \mathscr{L}_1) \sim (\psi_2{}^s, \mathscr{L}_2)$.

We can now state

*Theorem* 4.6. Let $(A, \mathscr{L})$ be a representation of a group $G$ in $\mathscr{L}$. Assume that $A$ is fully decomposable, i.e., $\mathscr{L} = \oplus_\alpha \mathscr{M}_\alpha$ where each $\mathscr{M}_\alpha$ is invariant under $A$ and is an $A$-irreducible subcroc of $\mathscr{L}$. Then the decomposition $\mathscr{L} = \oplus_\alpha \mathscr{M}_\alpha$ is unique except for order and equivalence.

*Proof.* Let $\mathscr{L} = \oplus_\alpha \mathscr{M}_\alpha$. From the commutation law in Def. 9 we have for each $A(g)$

$$\phi A(g) = \{A_\alpha(g)\}\phi$$

i.e., we have the equivalence

$$(A, \mathscr{L}) \sim (\{A_\alpha\}, \bigoplus_\alpha \mathscr{M}_\alpha)$$

Let now $\hat{x}_\beta$ be defined as follows:

$$\hat{x}_\beta = \{y\}_\alpha$$

where

$$y_\alpha = \begin{cases} x_\alpha & \text{if } \alpha = \beta \\ a_\alpha & \text{if } \alpha \neq \beta \end{cases}$$

Consider the subcroc $\hat{\mathscr{M}}_\alpha \subset \mathrm{V}_\alpha \mathscr{M}_\alpha$ which is formed by all $\hat{x}_\alpha$ for any fixed but arbitrary $\alpha$. Then $\hat{\mathscr{M}}_\alpha$ is a segment in $\mathrm{V}_\alpha \mathscr{M}_\alpha$. Therefore $\phi^{-1}\hat{\mathscr{M}}_\alpha$ is a segment in $\mathscr{L}$. It is also easily seen that any $x \in \mathscr{L}$ can be expressed as an orthogonal union, i.e.,

$$x = \bigvee_\alpha x_\alpha$$

where

$$x_\alpha \in \phi^{-1}\hat{\mathscr{M}}_\alpha \text{ and } x_\alpha < x'_\beta \; (\alpha \neq \beta)$$

Furthermore, because of the equivalence stated at the beginning of the proof, one sees that each segment $\phi^{-1}\hat{\mathscr{M}}_\alpha$ is irreducible under $A$ in $\mathscr{L}$.
Now suppose that

$$\mathscr{L} = \bigoplus_\alpha \mathscr{M}_\alpha \; (\alpha \in K) \qquad \text{and } \mathscr{L} = \bigoplus_\lambda \mathscr{N}_\lambda \; (\lambda \in L)$$

where $\mathscr{M}_\alpha$ and $\mathscr{N}_\lambda$ are $A$-irreducible subcrocs. Then, for any arbitrary $n_\lambda \in \phi^{-1}_{\mathscr{N}} \mathscr{N}_\lambda$,

$$n_\lambda = \bigvee_\alpha n_\lambda{}^\alpha$$

where

$$n_\lambda{}^\alpha \in \phi^{-1}_{\mathscr{M}} \hat{\mathscr{M}}_\alpha$$

For a fixed $\alpha$, the set $\{n_\lambda{}^\alpha\}$ forms a subcroc in $\phi^{-1}_{\mathscr{M}}\hat{\mathscr{M}}_\alpha$, which is invariant under $A$, because $\phi^{-1}_{\mathscr{N}}\hat{\mathscr{N}}_\lambda$ is such. But this contradicts the assumed $A$-irreducibility of $\phi^{-1}_{\mathscr{M}}\hat{\mathscr{M}}_\alpha$, unless that subcroc is $\{0\}$, $\{0, b_\alpha\}$, or $\phi^{-1}_{\mathscr{M}}\hat{\mathscr{M}}_\alpha$ itself.

Furthermore, since $\phi^{-1}\hat{\mathscr{N}}_\lambda$ is an $A$-irreducible segment, only one component is different from 0, which we shall call $\phi^{-1}_{\mathscr{M}}\hat{\mathscr{M}}_\alpha$. Clearly, we have the identity isomorphism

$$\phi^{-1}_{\mathscr{M}}\hat{\mathscr{M}}_\alpha = \phi^{-1}_{\mathscr{N}}\hat{\mathscr{N}}_\lambda$$

with some $\alpha \in K$, and some $\lambda \in L$. Therefore, $\phi^{-1}\hat{\mathscr{M}}_\alpha$ is equivalent to $\phi^{-1}_{\mathscr{N}}\hat{\mathscr{N}}_\alpha$. However, the mapping

$$\hat{\mathscr{M}}_\alpha \to \mathscr{M}_\alpha$$

given by

$$\hat{x}_\alpha \mapsto x_\alpha$$

reveals that $\mathscr{M}_\alpha$ is equivalent to $\hat{\mathscr{M}}_\alpha$. Since the $\mathscr{M}_\alpha$ and the $\mathscr{N}_\lambda$ play a dual role, the theorem now follows trivially.

*Remark*. The theorem still holds true if, instead of invariance under the group $A$, we demand only decomposition into $t$-irreducible and $t$-invariant subcrocs, where $t$ is an arbitrary given automorphism.

In summary, we see that our representation theorems correspond closely to those pertinent for representations in Hilbert spaces.

*Note Added in Manuscript.* After this work has been completed we became aware of a paper by Gudder (1971) and one by Gallone and Mania (1971), which treat closely related questions.

### *Appendix A: Proof of Lemma* 9

(a) Assume that $\mathscr{L}$ is $\phi$-isomorphic to $V_\alpha \mathscr{L}_\alpha$, where $\mathscr{L}_\alpha = [0, b_\alpha]$, $V_\alpha x_\alpha \equiv V_\alpha(b_\alpha \Lambda x) = x$, $\mathscr{L}_\alpha \perp \mathscr{L}_\beta$. Repeating the argument presented toward the end of the proof of Lemma 7, we see that $\phi x = \{x \Lambda b_\alpha\}$ If we define the family $\hat{b}_\alpha = \{x_j\}$ where

$$x_j = \begin{cases} b_\alpha & \text{if } j = \alpha \\ O & \text{if } j \neq \alpha \end{cases}$$

then we now see that $\phi b_\alpha = \hat{b}_\alpha$. Since $\hat{b}_\alpha$ is in the center of $V_\alpha \mathscr{L}_\alpha$ we have $b_\alpha$ in the center of $\mathscr{L}$. Let now $z \in \mathscr{C}(\mathscr{L})$, so that $\phi z = \{z_\alpha\} \in (V_\alpha \mathscr{L}_\alpha)$. Therefore, $z_\alpha \in \mathscr{C}(\mathscr{L}_\alpha)$. But, since each $\mathscr{L}_\alpha$ is $c$-irreducible, we have that either $z_\alpha = O$, or $z_\alpha = b_\alpha$. Consequently, $z = V_\lambda b_\lambda$.with $\lambda \in L$ where the index set $L$ is a subset of the index set $A$ from which the $\alpha$ are taken. Hence, in $\mathscr{C}(\mathscr{L})$, each $b_\alpha$ covers $O$, i.e., it is an atom (see Piron, 1964). Furthermore, since every $z \in \mathscr{C}(\mathscr{L})$ is some supremum of elements $b_\alpha$, these are the only atoms in $\mathscr{C}(\mathscr{L})$. Hence $\mathscr{C}(\mathscr{L})$ is atomic with atoms $b_\alpha$, and $b_\alpha \perp b_\beta$ (if $\beta \neq \alpha$). All that is left to show is that the covering law is obeyed in $\mathscr{C}(\mathscr{L})$. Suppose that for some $\sigma \in A$, there exists $x, y \in \mathscr{C}(\mathscr{L})$ such that

$$x < y < x V b_\sigma$$

Then we have

$$x = \underset{\delta \in D}{V} b_\delta, \qquad y = \underset{\rho \in R}{V} y_\rho$$

where $R \subset A, D \subset A$. Further, since $x < y$, we have $D \subset R$. In addition, because of the assumption $y < x V b_\sigma$, we have $R \subset D \cup \{\sigma\}$. Therefore,

$$D \cup \{\sigma\} \subset R \cup \{\sigma\} \subset D \cup \{\sigma\}$$

i.e., $R \cup \{\sigma\} = D \cup \{\sigma\}$. Thus, either we have $R = D \cup \{\sigma\}$ or $R = D$ and correspondingly either $y = x V b_\sigma$ or $y = x$. Thus, the covering law holds, which concludes the proof of the first part of the lemma.

(b) Assume that $\mathscr{C}(\mathscr{L})$ is a Boolean atomic CROC obeying the covering law and such that its atoms $b_\alpha (\alpha \in A)$ are disjoint, i.e., $b_\alpha \perp b_\beta$ for $\alpha \neq \beta$. As shown in Piron (1964), we have $I = \mathrm{V}_\alpha b_\alpha$. Therefore, for any $x \in \mathscr{L}$,

$$\underset{\alpha}{\mathrm{V}}(x \wedge b_\alpha) \equiv \underset{\alpha}{\mathrm{V}} x_\alpha = [x \wedge (\underset{\alpha}{\mathrm{V}} b_\alpha)] = x \tag{A1}$$

Also,

$$x'_\alpha = x' \mathrm{V} b'_\alpha > b_\beta > b_\beta \wedge x = x_\beta \tag{A2}$$

for all $\alpha \neq \beta$. Consider now the segments $\mathscr{L}_\alpha \equiv [0, b_\alpha]$. From (A2) we see that $\mathscr{L}_\alpha \perp \mathscr{L}_\beta$. Then we note that, because of (A1) and (A2), the map

$$\phi : \mathscr{L} \to \underset{\alpha}{\mathrm{V}} \mathscr{L}_\alpha$$

given by

$$\phi x = \{x_\alpha\}$$

is trivially an isomorphism.

Suppose now that $z_\alpha \in \mathscr{C}(\mathscr{L}_\alpha)$ but $z_\alpha \neq 0$, $z_\alpha \neq b_\alpha$, i.e., assume that $\mathscr{L}_\alpha$ is $c$-reducible. Then

$$(z_\alpha \wedge x_\alpha^{*\alpha}) \mathrm{V} x_\alpha > z_\alpha \tag{A3}$$

However, $x_\alpha^{*\alpha} = x'_\alpha \wedge b_\alpha = (x' \mathrm{V} b'_\alpha) \wedge b_\alpha = x' \wedge b_\alpha$, so that, by (A3),

$$(z_\alpha \wedge x_\alpha^{*\alpha}) \mathrm{V} x_\alpha = (z_\alpha \wedge x' \wedge b_\alpha) \mathrm{V} (b_\alpha \wedge x) = (z_\alpha \wedge x') \mathrm{V} (b_\alpha \wedge x) > z_\alpha$$

But, since $x > x \wedge b_\alpha$, we get $(z_\alpha \wedge x') \mathrm{V} x > (z_\alpha \wedge x') \mathrm{V} (b_\alpha \wedge x) > z_\alpha$. This means that $z_\alpha \leftrightarrow x$ for any $x$, hence $z_\alpha \in \mathscr{C}(\mathscr{L})$. But then one could write $z_\alpha = \mathrm{V}_\beta b_\beta$ which contradicts the assumption regarding $z_\alpha$. Consequently, each $\mathscr{L}_\alpha$ must be $c$-irreducible, and this concludes the proof.

## *Appendix B: Example of a Direct Sum Decomposition*

Let $\mathscr{L}$ be the non-Boolean CROC specified by the order diagram of Figure 1. Define

$$\mathscr{M}_1 = \{0, a, b, c, d, a', b', c', d', I\}$$

This is trivially a subcroc of $\mathscr{L}$.
Define

$$\mathscr{M}_2 = \{0, f\} = [0, f]$$

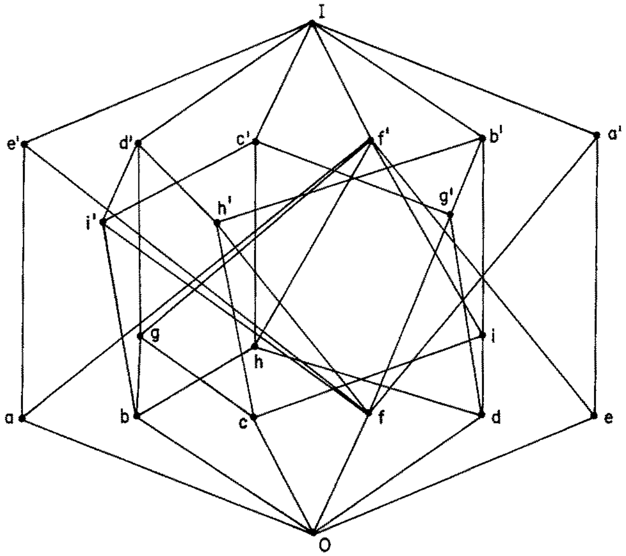


Figure 1–A non-Boolean CROC.

Consider now the map of $\phi : \mathscr{L} \rightarrow \mathscr{M}_1 \mathrm{V} \mathscr{M}_2$ given by the following assignments:

| $x$ | $0$ | $a$ | $b$ | $c$ | $d$ | $e$ | $f$ |
|---|---|---|---|---|---|---|---|
| $\phi x$ | $(0, 0)$ | $(a, 0)$ | $(b, 0)$ | $(c, 0)$ | $(d, 0)$ | $(a', 0)$ | $(0, f)$ |

| $x$ | $g$ | $h$ | $i$ | $a'$ | $b'$ | $c'$ |
|---|---|---|---|---|---|---|
| $\phi x$ | $(d', 0)$ | $(c', 0)$ | $(b', 0)$ | $(a', f)$ | $(b', f)$ | $(c', f)$ |

| $x$ | $d'$ | $e'$ | $f'$ | $g'$ | $h'$ | $i'$ | $I$ |
|---|---|---|---|---|---|---|---|
| $\phi x$ | $(d', f)$ | $(a, f)$ | $(I, 0)$ | $(d, f)$ | $(c, f)$ | $(b, f)$ | $(I, f)$ |

By a straightforward but somewhat lengthy calculation it can be checked that $\phi$ is an isomorphism and that it obeys the commutation law of Def. 9. Hence, we established the direct sum deomposition

$$\mathscr{L} = \mathscr{M}_1 \oplus \mathscr{M}_2$$

Observe that $\mathscr{M}_1$ is not a segment and that $\mathscr{M}_1$ is not orthogonal to $\mathscr{M}_2$.

### *Appendix C: Some Properties of Linear Automorphisms*

In the following, $T$ will denote a linear automorphism and fixed points will be denoted by $x_\alpha$; arbitrary elements of $\mathscr{L}$ will be written $y$, $z$, etc. The abbreviation fp will be used for "fixed point."

*Theorem* C1. If $x_0$ is a fp of $T$, so is $x_0'$.

*Proof*: $Tx_0' = (Tx_0)' = x_0'$.

*Theorem* C2. If $x_\alpha$ ($\alpha \in A$) are fp's of $T$, so are all suprema and infima.

*Proof*. $T(\mathrm{V}_\alpha x_\alpha) = \mathrm{V}_\alpha Tx_\alpha = \mathrm{V}x$, and similarly for $\Lambda_\alpha$.

*Theorem* C3. If $T$ is linear, so is $T^{-1}$ and has the same fp's.

*Proof*. $Tx_0 = x_0$ implies $x_0 = T^{-1}x_0$.

*Theorem* C4. If $y \perp x_0$ then $Ty \perp x_0$ and conversely.

*Proof*. (a) $y < x_0'$ implies $Ty < Tx_0' = x_0'$.

(b) $Ty < x_0'$ implies $Ty < Tx_0'$ and thus $y < x_0'$.

*Corollary to Theorem* C4. The set $K = \{y_\alpha \mid y_\alpha \perp x_0\}$ is an invariant subcroc.

*Proof*. Invariance under $T$ follows trivially from the theorem. To show that $K$ is a subcroc, observe that $y_\beta < x_0'$ (for $\beta \in B$) implies $\Lambda_\beta y_\beta < x_0'$ and also $\mathrm{V}_\beta y_\beta < x_0'$. Also, $0 \in K$. To show closure under the star map, note that for any $y_\beta' \in K$, $y_\beta' \Lambda x_0' < x_0'$ so that $y_\beta^* = y_\beta' \Lambda x_0' \in K$.

*Remark*. This corollary holds also for the set $K' = \{v_\alpha \mid v_\alpha \perp x_0'\}$.

*Theorem* C5. For any fp, $x_0 \leftrightarrow y$ iff $x_0 \leftrightarrow Ty$.

*Proof*.

$$x_0 \leftrightarrow y \Leftrightarrow x_0 \Lambda (y \mathrm{V} x_0') < y$$

$$\Leftrightarrow T[x_0 \Lambda (y \mathrm{V} x_0')] < Ty \Leftrightarrow Tx_0 \Lambda (Ty \mathrm{V} Tx_0') < Ty$$

$$\Leftrightarrow x_0 \Lambda (Ty \mathrm{V} x_0') < Ty \Leftrightarrow x_0 \leftrightarrow Ty.$$

Here we used Def. 3, part (ii), and Theorem C1.

*Corollary to Theorem* C5. The set $H = \{y_\alpha \mid y_\alpha \leftrightarrow x_0\}$ is an invariant subcroc.

*Proof*. Invariance under $T$ is trivial from the theorem. To show that we have a subcroc, observe that suprema and infima of families from $H$ are also compatible with $x_0$ and that $0 \in \mathrm{H}$, $\mathrm{I} \in \mathrm{H}$, so that $y_\alpha^* = y_\alpha'$. Also, $y_\alpha' \leftrightarrow x_0$, hence $y_\alpha^* \in \mathrm{H}$.

*Theorem* C6. If $T_1T_2 = T_2T_1$ and $x_0$ is a fp of $T_2$, then $T_1x_0$ is a fp of $T_2$.

*Proof*. From the assumptions, $T_1x_0 = T_2(T_1x_0)$ trivially follows.

*Theorem* C7. If $T_1$ and $T_2$ have a common fp $x_0$, then $T_1T_2$, $T_2T_1$, $T_1^{-1}T_2$, $T_2^{-1}T_1$, $T_1^{-1}T_2^{-1}$, $T_2^{-1}T_1^{-1}$ are also linear and have the fp $x_0$.

*Proof.* $T_1T_2x_0 = T_1x_0 = x_0$, and similarly for $T_2T_1$. Then note that $T_1^{-1}$ and $T_2^{-1}$ also have fp $x_0$.

*Theorem* C8. In an atomic (cf. Piron, 1969) CROC, if all atoms are fp's of some $T$, then $T$ is the trivial automorphism.

*Proof.* In an atomic CROC, any $y \in \mathscr{L}$ can be written as a supremum of some collection of atoms. By the assumption of this theorem and because of Theorem C2, we then have $Ty = y$ for all $y \in \mathscr{L}$.

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